

FULLY-PROJECTED SUBSETS

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ABSTRACT. Let k and i_1, \dots, i_n be natural numbers. Place k balls into a multidimensional box of $i_1 \times \dots \times i_n$ cells, no more than one ball to each cell, such that the projections to each of the coordinate axes have cardinalities i_1, \dots, i_n , respectively. We generalize earlier work of Wang, Lee, and Tan to find a formula for the alternating sum of the number of these fully-projected subsets.

1. INTRODUCTION

Let k and i_1, \dots, i_n be natural numbers. We consider the task of placing k balls into a multidimensional box of $i_1 \times \dots \times i_n$ cells, such that each cell contains at most one ball, and such that each projection to each coordinate has cardinality i_1, \dots, i_n , respectively. This generalizes work of Wang, Lee, and Tan [2] on the two-dimensional version of the problem, where the condition requires that each row and each column contains at least one ball. Some of their interest in the formula stemmed from its role in the theory of falling random subsets in fuzzy statistics.

If we call such subsets fully-projected, then, generalizing the result of [2], we have the following formula involving the alternating sum of the numbers t_k of these subsets.

Theorem 1. *Let k and i_1, \dots, i_n be natural numbers, and, for $j = 1, \dots, n$, let $I_j = \{1, \dots, i_j\}$. If $t_k = t_k(i_1, \dots, i_n)$ denotes the number of all fully-projected k -subsets of $\prod_{j=1}^n I_j$, then*

$$(1) \quad \sum_{k=1}^{i_1 \cdots i_n} (-1)^{k-1} t_k = (-1)^{i_1 + \dots + i_n}.$$

Our proof of Theorem 1 follows the approach of Wang, Lee, and Tan. The combinatorial analysis here, provided in Section 2 below, requires a small bit of care in order to avoid a blurred forest of unions and intersections over the index sets and elements.

Work of Fulmek [1] generalized the result of Wang, Lee, and Tan in a different direction, leading to an interpretation of the formula in the language of dual rook polynomials. The rook polynomial of a board \mathfrak{B} (an arbitrary subset of the cells of an $m \times n$ array) is defined by

$$(2) \quad P_{\mathfrak{B}}(x) = \sum_{k \geq 0} R_k(\mathfrak{B}) x^k,$$

where $R_k(\mathfrak{B})$ is the number of ways to place k non-attacking rooks on the board \mathfrak{B} . The property of non-attacking can be viewed as the requirement that each row and

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each column contains *at most one* rook. Fulmek considered a sort of dual notion. Letting $\tilde{R}_k(\mathfrak{B})$ denote the number of ways to place k rooks on \mathfrak{B} such that each row and each column contains *at least one* rook, Fulmek called the polynomial

$$(3) \quad \tilde{P}_{\mathfrak{B}}(x) = \sum_{k \geq 0} \tilde{R}_k(\mathfrak{B}) x^k$$

the dual rook polynomial of \mathfrak{B} . A key result from [1], generalizing the Wang, Lee, and Tan formula, gives that $\tilde{P}_{\mathfrak{B}}(-1)$ is always -1 , 0 , or 1 for skew Ferrers boards.

Fulmek's paper also contains some interesting conjectures related to these matters, including, e.g., the question of the log-concavity of the dual rook numbers $\tilde{R}_k(\mathfrak{B})$. The resolution of those conjectures in their original formulation (or the consideration of appropriate multidimensional generalizations) and the finer combinatorial and statistical properties of the numbers t_k present multiple avenues for further work.

2. COUNTING VIA INCLUSION-EXCLUSION

To aid in the combinatorial analysis, we begin with a definition of fully-projected subset that clarifies the projection property. The proof of Theorem 1 appears following this definition.

Definition (Fully-projected k -subset). *Let $S_k \subseteq \prod_{j=1}^n I_j$ be a k -element subset of $\prod_{j=1}^n I_j$. Call S_k a fully-projected subset of $\prod_{j=1}^n I_j$, denoted by $S_k \hookrightarrow \prod_{j=1}^n I_j$, provided that, for $j = 1, \dots, n$, the set S_k satisfies*

$$(4) \quad \pi_j(S_k) = I_j = \{1, \dots, i_j\}.$$

Here π_j denotes projection onto the j th coordinate, so that $\pi_j(a_1, \dots, a_n) = a_j$.

Proof of Theorem 1. Let Q denote the set of all k -subsets of $M = \prod_{j=1}^n I_j$, and let A denote the set of all fully-projected k -subsets of M . Further, for $j = 1, \dots, n$ and $r = 1, \dots, i_j$, let $B_{j,r}$ denote the set of k -element subsets of M that avoid element r within coordinate j . Succinctly, we have

$$(5) \quad \begin{aligned} Q &= \{S_k : S_k \subseteq M\}, \\ A &= \{S_k : S_k \hookrightarrow M\}, \\ B_{j,r} &= \{S_k : S_k \subseteq I_1 \times \dots \times (I_j \setminus \{r\}) \times \dots \times I_n\}. \end{aligned}$$

Note that $|Q| = \binom{i_1 \dots i_n}{k}$. Also, from the above, we see that

$$(6) \quad A = Q \setminus \left(\bigcup_{j=1}^n \bigcup_{r=1}^{i_j} B_{j,r} \right),$$

and

$$(7) \quad t_k = |A| = |Q| - \left| \bigcup_{j=1}^n \bigcup_{r=1}^{i_j} B_{j,r} \right|,$$

because the fully-projected k -element subsets collected in A are exactly the k -element subsets that, together, miss no element in any coordinate.

Define α by

$$(8) \quad \alpha = \left| \bigcup_{j=1}^n \bigcup_{r=1}^{i_j} B_{j,r} \right|.$$

Then, by the inclusion-exclusion principle, letting the index sets J_j range over subsets of $I_j = \{1, \dots, i_j\}$ and using \sum' to indicate a sum that excludes the case $m_1 = \dots = m_n = 0$, we have that

$$\begin{aligned} \alpha &= \sum'_{\substack{0 \leq m_j \leq i_j \\ \text{for } j=1, \dots, n}} (-1)^{m_1 + \dots + m_n - 1} \sum_{\substack{J_1 \subseteq I_1, \dots, J_n \subseteq I_n \\ |J_1|=m_1, \dots, |J_n|=m_n}} \left| \bigcap_{j=1}^n \bigcap_{r_j \in J_j} B_{j,r_j} \right| \\ &= \sum'_{\substack{0 \leq m_j \leq i_j \\ \text{for } j=1, \dots, n}} (-1)^{m_1 + \dots + m_n - 1} \sum_{\substack{J_1 \subseteq I_1, \dots, J_n \subseteq I_n \\ |J_1|=m_1, \dots, |J_n|=m_n}} \binom{(i_1 - m_1) \cdots (i_n - m_n)}{k} \\ &= \sum'_{\substack{0 \leq m_j \leq i_j \\ \text{for } j=1, \dots, n}} (-1)^{m_1 + \dots + m_n - 1} \binom{i_1}{m_1} \cdots \binom{i_n}{m_n} \binom{(i_1 - m_1) \cdots (i_n - m_n)}{k}. \end{aligned}$$

We have then, by (7), (8), and the above expression for α , that

$$(9) \quad t_k = |Q| - \alpha$$

$$(10) \quad = \binom{i_1 \cdots i_n}{k} - \alpha$$

$$(11) \quad = \sum_{\substack{0 \leq m_j \leq i_j \\ \text{for } j=1, \dots, n}} (-1)^{m_1 + \dots + m_n} \binom{i_1}{m_1} \cdots \binom{i_n}{m_n} \binom{(i_1 - m_1) \cdots (i_n - m_n)}{k}.$$

Using (11), we obtain that

$$\begin{aligned}
& \sum_{k=1}^{i_1 \cdots i_n} (-1)^{k-1} t_k \\
&= \sum_{k=1}^{i_1 \cdots i_n} (-1)^{k-1} \sum_{\substack{0 \leq m_j \leq i_j \\ \text{for } j=1, \dots, n}} (-1)^{m_1 + \cdots + m_n} \binom{i_1}{m_1} \cdots \binom{i_n}{m_n} \binom{(i_1 - m_1) \cdots (i_n - m_n)}{k} \\
&= \sum_{\substack{0 \leq m_j \leq i_j \\ \text{for } j=1, \dots, n}} (-1)^{m_1 + \cdots + m_n} \binom{i_1}{m_1} \cdots \binom{i_n}{m_n} \sum_{k=1}^{i_1 \cdots i_n} (-1)^{k-1} \binom{(i_1 - m_1) \cdots (i_n - m_n)}{k} \\
&= \sum_{\substack{0 \leq m_j < i_j \\ \text{for } j=1, \dots, n}} (-1)^{m_1 + \cdots + m_n} \binom{i_1}{m_1} \cdots \binom{i_n}{m_n} \sum_{k=1}^{i_1 \cdots i_n} (-1)^{k-1} \binom{(i_1 - m_1) \cdots (i_n - m_n)}{k} \\
&= \sum_{\substack{0 \leq m_j < i_j \\ \text{for } j=1, \dots, n}} (-1)^{m_1 + \cdots + m_n} \binom{i_1}{m_1} \cdots \binom{i_n}{m_n} \cdot 1 \\
&= \prod_{j=1}^n \left(\sum_{0 \leq m_j < i_j} (-1)^{m_j} \binom{i_j}{m_j} \right) \\
&= (-1)^{i_1 + \cdots + i_n},
\end{aligned}$$

which completes the proof of Theorem 1. \square

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